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Eigenvalue density of correlated complex random Wishart matrices

Steven H. Simon and Aris L. Moustakas
Lucent Technologies, Bell Labs, Murray Hill, New Jersey 07974, USA
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Using a character expansion method, we calculate exactly the eigenvalue density of random matrices of the form $\mathbf{M}^{\dagger}\mathbf{M}$ where $\mathbf{M}$ is a complex matrix drawn from a normalized distribution $P(\mathbf{M}) \sim \exp(-\mathrm{Tr}\{\mathbf{AMBM}^{\dagger}\})$ with $\mathbf{A}$ and $\mathbf{B}$ positive definite (square) matrices of arbitrary dimensions. Such so-called correlated Wishart matrices occur in many fields ranging from information theory to multivariate analysis.

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Physicists usually think of Wigner and Dyson as the fathers of random matrix theory [1,2]. However, 20 years before their first work on the subject, Wishart [3] examined random matrices of the form $\mathbf{MM}^{\dagger}$ as a tool for studying multivariate data. The properties of these so-called Wishart matrices, which are viewed as "fundamental to multivariate statistical analysis" [4], also find important applications in fields like information theory and communications [5–7], mesoscopics [8], high energy physics [9], and econophysics [10].

In many cases one is interested in Wishart matrices where the elements of $\mathbf{M}$ are not completely independent random variables, but have correlations along rows and/or columns. Important examples of this case occur in data analysis problems [11], where random samples have temporal and spatial correlations, and particularly in wireless communication and information theory [5,6]. The purpose of this paper is to derive the eigenvalue density of correlated complex Wishart matrices exactly. This problem has been solved previously (and applications discussed) in the limit of large matrices [6,11]. However, there are many situations where one is explicitly interested in matrices of finite size (or even small size) [5–8], which we address in the current paper. We note that if either $\mathbf{A}$ or $\mathbf{B}$ is proportional to unity, simpler techniques can be used [12].

We first define the problem more precisely. Let $\mathbf{M}$ be an $N\times N'$ complex matrix chosen from a normalized distribution

$$P(\mathbf{M}) = \pi^{-NN'}\mathcal{N}\exp(-\mathrm{Tr}\{\mathbf{AMBM}^{\dagger}\}) \tag{1}$$

with $\mathbf{A}$ and $\mathbf{B}$ positive definite square matrices that define the correlations, and Tr indicating the trace. Here, $\mathcal{N}^{-1} = \det[\mathbf{A}]^{N'}\det[\mathbf{B}]^{N}$ and the factors of π are normalization constants. An equivalent definition would be to let $\mathbf{M} = \mathbf{A}^{-1/2}\mathbf{Z}\,\mathbf{B}^{-1/2}$ where $\mathbf{Z}$ is a random complex matrix with independent entries of zero mean and unit covariance. Note that $\mathbf{A}$ is $N\times N$ and $\mathbf{B}$ is $N'\times N'$. Without loss of generality, we can assume $N\geqslant N'$. For any operator $O(\mathbf{M})$ we define the expectation bracket $\langle O\rangle$ to be an average over realizations of $\mathbf{M}$ so that $\langle O\rangle \equiv \int d\mathbf{M}O(\mathbf{M})P(\mathbf{M})$. Note that the normalization is such that $\langle 1\rangle = 1$.

Let λ_n be the N' eigenvalues of $\mathbf{M}^{\dagger}\mathbf{M}$ or equivalently the N' nonzero eigenvalues of $\mathbf{MM}^{\dagger}$ (we will also have $N-N'$ eigenvalues of $\mathbf{MM}^{\dagger}$ precisely zero). We define the following quantities to calculate:

$$G_{\nu}(z) = \left\langle \prod_{n=1}^{N'} (\lambda_n - z)^{\nu} \right\rangle = \langle \det(\mathbf{M}^{\dagger}\mathbf{M} - z)^{\nu}\rangle, \tag{2}$$

$$H(z) = \partial G_{\nu}(z)/\partial \nu_{\nu=0} = \left\langle \sum_{n=1}^{N'} \ln(\lambda_n - z) \right\rangle, \tag{3}$$

$$C(\lambda) = \lim_{\epsilon\to 0}[H(\lambda - i\epsilon) - H(\lambda + i\epsilon)]/2\pi i, \tag{4}$$

$$= \left\langle \sum_{n=1}^{N'} \theta(\lambda - \lambda_n) \right\rangle = \int_{-\infty}^{\lambda} dx\ \rho(x), \tag{5}$$

$$\rho(\lambda) = dC(\lambda)/d\lambda = \left\langle \sum_{n=1}^{N'} \delta(\lambda - \lambda_n) \right\rangle, \tag{6}$$

where θ is the step function, λ is assumed real, and in going from Eq. (4) to Eq. (5) we have used $\lim_{\epsilon\to 0} \mathrm{Im}\ln(-y+i\epsilon) = \pi\theta(y)$, which is true for real y. The quantity we are most interested in is the eigenvalue density $\rho(\lambda)$. From Eqs. (2)–(6) it is clear that we can obtain ρ by calculating $G_{\nu}(z)$.

Below, we will show that

$$G_{\nu}(z) = Q_{\nu}(z)R_{\nu}\det L_{ij}, \tag{7}$$

$$Q_{\nu}(z)^{-1} = \Delta_N(a)\Delta_{N'}(b)(-z)^{N'(N'-1)/2}J_{\nu}, \tag{8}$$

$$J_{\nu} = \prod_{i=1}^{N-1} (\nu + i)^{i}, \tag{9}$$

$$R_{\nu} = \prod_{j=1}^{N-N'-1} (N + \nu - j)^{N-N'-j}, \tag{10}$$

where R_{ν} is defined to be unity for $N'\geqslant N-1$. In Eq. (7), L_{ij} is an $N\times N$ matrix with elements $L_{ij} = g(a_i b_j; \nu+N, z)$ for $j \leqslant N'$ and $L_{ij} = a_i^{j-1}$ for $j > N'$, where we have defined a_i and b_j to be the eigenvalues of the matrices $\mathbf{A}$ and $\mathbf{B}$. The function g is given by

$$g(x;\alpha,z) = x^{N-\alpha}e^{-zx}\Gamma(\alpha, -zx) \tag{11}$$

$$=x^N\int_0^\infty d\lambda(\lambda-z)^{\alpha-1}e^{-x\lambda} \qquad (12)$$

with Γ the incomplete gamma function [14], and we note that for integer $\alpha>0$ we have the simple form

$$g(x,\alpha,z)=x^{N-\alpha}(\alpha-1)!\sum_{m=0}^{\alpha-1}(-zx)^m/m! \qquad (13)$$

In Eq. (8) and throughout this paper we use the notation

$$\Delta_V(x)=\prod_{1\le i<j\le V}(x_j-x_i)=\det[x_j^{i-1}] \qquad (14)$$

to represent a V-dimensional Vandermonde determinant.

From $G_\nu(z)$ we calculate $C(\lambda)$ using Eqs. (3) and (4). The differentiation [Eq. (3)] with respect to ν brings down a factor of $\ln(\lambda-z)$ in the argument of Eq. (12). This logarithm becomes a step function when the limit is taken in Eq. (4). We obtain

$$C(\lambda)=N'-Q_0(\lambda)R_0\sum_{n=1}^{N}\det K_{ij}^{(n)}, \qquad (15)$$

where we also used $G_{\nu=0}(z)\equiv 1$. Here, we have defined $N\times N$ matrices $K^{(n)}$ with $K_{ij}^{(n)}=g(a_ib_j;N,\lambda)$ for $n\neq i$ and $j\le N'$ and $K_{ij}^{(n)}=a_i^{j-1}$ for $j>N'$ and $n\neq i$. For the case of $i=n$ we have $K_{nj}^{(n)}=e^{-a_nb_j\lambda}(N-1)!$ for $j\le N'$ and $K_{nj}^{(n)}=0$ for $j>N'$.

We then differentiate Eq. (15) [see Eq. (6)] to obtain

$$\rho(\lambda)=Q_0(\lambda)R_0\sum_{n=1}^{N}\left[\det\tilde{K}_{ij}^{(n)}+\sum_{m=1,m\neq n}^{N}\det T_{ij}^{(nm)}\right], \qquad (16)$$

where $\tilde{K}^{(n)}$ and $T^{(nm)}$ are $N\times N$ matrices with elements defined as follows: $\tilde{K}_{ij}^{(n)}=K_{ij}^{(n)}$ for $i\neq n$ and $\tilde{K}_{nj}^{(n)}=\{a_nb_j+[N'(N'-1)/2]/\lambda\}K_{nj}^{(n)}$ for $j\le N'$ and $\tilde{K}_{nj}^{(n)}=0$ for $j>N'$. Also $T_{ij}^{(nm)}=K_{ij}^{(n)}$ for $i\neq m$ with $T_{mj}^{(nm)}=(N-1)a_mb_j\, g(a_mb_j,N-1,\lambda)$ for $j\le N'$ and $T_{mj}^{(nm)}=0$ for $j>N'$.

The above expressions are our main results. The remainder of this paper comprises the proof of Eqs. (7)–(11) from which all of our other results follow. We start by focusing on the case of square matrices $\mathbf{M}$ (so $N=N'$). We write

$$G_\nu(z)=\mathcal{N}\pi^{-N^2}\int d\mathbf{M}\; e^{-\mathrm{Tr}\{\mathbf{AMBM}^\dagger\}}\det(\mathbf{MM}^\dagger-z)^\nu.$$

We then define $\mathbf{M}=\mathbf{UmV}$ where $\mathbf{m}$ is a diagonal matrix of the singular values m_i of $\mathbf{M}$ and $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices. We then separate the integral over $\mathbf{M}$ into integrals over the eigenvalues $\lambda_i=|m_i|^2$ of $\mathbf{MM}^\dagger$ and "angular" integrals over $\mathbf{U}$ and $\mathbf{V}$. This approach, common in random matrix theory[1], yields

$$G_\nu(z)=\mathcal{CN}\int d\boldsymbol{\lambda}\prod_{j=1}^{N}(\lambda_j-z)^\nu\Delta_N(\lambda)^2D_{\mathbf{A},\mathbf{B}}(\boldsymbol{\lambda}), \qquad (17)$$

where $\boldsymbol{\lambda}=\mathbf{mm}^\dagger$ is the diagonal matrix of eigenvalues λ_i, and $\int d\boldsymbol{\lambda}=\prod_{i=1}^N\int_0^\infty d\lambda_i$, and $\mathcal{C}$ is an N-dependent numerical constant (which we will not keep track of explicitly but will fix at the end of the calculation). Here, $\Delta_N(\lambda)^2$ is the Vandermonde determinant squared of the λ_i's (which is the Jacobian of the transformation) and

$$D_{\mathbf{A},\mathbf{B}}(\boldsymbol{\lambda})=\int_{U(N)}d\mathbf{U}\int_{U(N)}d\mathbf{V}\; e^{-\mathrm{Tr}\{\mathbf{AUmVBV}^\dagger\mathbf{m}^\dagger\mathbf{U}^\dagger\}},$$

where $\mathbf{U}$ and $\mathbf{V}$ are $N\times N$ unitary matrices which are integrated with the usual Haar measure over $U(N)$. Note that we have written the integral $D_{\mathbf{A},\mathbf{B}}$ as a function of $\boldsymbol{\lambda}=\mathbf{mm}^\dagger$ (we will see that this is indeed true). Here $\mathcal{CN}\Delta_N(\lambda)^2D_{\mathbf{A},\mathbf{B}}(\boldsymbol{\lambda})$ is precisely the joint probability density of the λ's. As such, it is clear that this density (and also D) must vanish exponentially if any of the λ's is taken to infinity (this will be important below). In particular, when $\mathbf{A}$, $\mathbf{B}$ are identity matrices we can see that $D_{\mathbf{A},\mathbf{B}}(\boldsymbol{\lambda})\sim\exp(-\Sigma_i\lambda_i)$.

To address these integrals over $U(N)$, we use the character expansion method discussed in depth in Ref. [13]. This allows us to write

$$e^{-\mathrm{Tr}\{\mathbf{AUmVBV}^\dagger\mathbf{m}^\dagger\mathbf{U}^\dagger\}}=\sum_r\alpha_r\chi_r(\mathbf{AUmVBV}^\dagger\mathbf{m}^\dagger\mathbf{U}^\dagger),$$

where α_r are expansion coefficients (discussed below), the sum is over representations r of $Gl(N)$, and χ_r is the character of the group element in the proper representation. [Note that the representation theory of $Gl(N)$ is identical to that of $U(N)$.] A character is just the trace taken in the proper representation, so we have

$$\chi_r(\mathbf{AUmVBV}^\dagger\mathbf{m}^\dagger\mathbf{U}^\dagger)=A_{ab}^rU_{bc}^rm_{cd}^rV_{de}^rB_{ef}^rV_{gf}^{r*}m_{hg}^{r*}U_{ah}^{r*}$$

with lower repeated indices summed (while the superscripts r tell us that the matrix is in representation r). We now use the orthogonality property [13]

$$\int_{U(N)}d\mathbf{U}\;U_{ab}^rU_{cd}^{r*}=d_r^{-1}\delta_{ac}\delta_{bd}$$

with d_r the dimension of representation r (discussed below). Combining the above three equations we obtain

$$D_{\mathbf{A},\mathbf{B}}(\boldsymbol{\lambda})=\sum_r\alpha_rd_r^{-2}\;\chi_r(\mathbf{A})\chi_r(\mathbf{B})\chi_r(\boldsymbol{\lambda}). \qquad (18)$$

As discussed in Ref. [13], each representation r is specified by a set of increasing integers $0\le k_N<k_{N-1}<\cdots<k_1$, so the sum written over r is actually an ordered sum over the k's ($k_j=N+n_j-j$ in the notation of Ref. [13]). In Ref. [13] it is also found that $\alpha_r=s(k)\det[1/(k_j+i-N)!]=s(k)\Delta_N(k)/C(k)$ where $C(k)=\prod_{j=1}^Nk_j!$. Here $s(k)=(-1)^\upsilon$ with $\upsilon=N(N-1)/2-\Sigma_j k_j$. In the same work [13] it is also shown that $\alpha_r/d_r=s(k)F_N/C(k)$ with $F_N=\prod_{j=1}^{N-1}j!$, from which we then obtain

$$\alpha_rd_r^{-2}=s(k)F_N^2/\Delta_N(k)C(k). \qquad (19)$$

The Weyl character formula tells us that [13]

$$\chi_r(\mathbf{X}) = \det[x_i^{k_j}]/\Delta_N(x) \tag{20}$$

with k_j the integers describing the representation r, and x_i the eigenvalues of $\mathbf{X}$. Substituting Eq. (20) into Eq. (18) we obtain

$$\Delta_N(\lambda)^2 D_{\mathbf{A},\mathbf{B}}(\boldsymbol{\lambda}) = \Delta_N(\lambda)\sum_r \Phi_r \det[\lambda_i^{k_j}], \tag{21}$$

$$\Phi_r = \alpha_r d_r^{-2}\chi_r(\mathbf{A})\chi_r(\mathbf{B}). \tag{22}$$

We next need the useful identity:

$$\Delta_N(\lambda) = \frac{1}{(-z)^{N(N-1)/2}}\det\left[\left(\frac{\lambda_i}{\lambda_i - z}\right)^{j-1}\right]\prod_{n=1}^{N}(\lambda_n - z)^{N-1}.$$

To show this we note that since $\Delta_N(\lambda)=\Pi_{i<j}(\lambda_i-\lambda_j)$ we can freely add a constant to each λ_i and not change Δ_N. Thus, we have $\Delta_N(\lambda)=\Delta_N(\lambda-z)$. Next we use $\Delta_N(x_1,\ldots,x_N) = \Delta_N(-1/x_1,\ldots,-1/x_N)\Pi_{j=1}^N x_j^{N-1}$ so that we can relate $\Delta_N(\lambda)$ to $\Delta_N(1/[z-\lambda])$. We then use $1/(z-\lambda)-1/z=\lambda/[z(z-\lambda)]$ and we again shift each term in the Vandermonde determinant by $-1/z$. Finally, we separate out factors of $-z$ and write the Vandermonde determinant as on the far right of Eq. (14).

Using the above expression for $\Delta_N(\lambda)$ and substituting Eq. (21) into Eq. (17) yields

$$G_\nu(z) = \frac{\mathcal{CN}}{(-z)^{N(N-1)/2}}\int d\lambda\prod_{i=1}^N(\lambda_i - z)^{\nu+N-1} \times\sum_r \Phi_r \det[\lambda_i^{k_j}]\det\left[\left(\frac{\lambda_i}{\lambda_i - z}\right)^{j-1}\right] \tag{23}$$

$$= \frac{\mathcal{CN}}{(-z)^{N(N-1)/2}}\sum_{c_1,\ldots,c_N}\epsilon_{c_1\ldots c_N}\int d\lambda\prod_{i=1}^N(\lambda_i - z)^{\nu+N-c_i} \times\left\{\left[\sum_r \Phi_r\sum_{d_1\cdots d_N}\epsilon_{d_1\cdots d_N}\prod_{i=1}^N\lambda_i^{k_{d_i}}\right]\prod_{i=1}^N\lambda_i^{c_i-1}\right\}. \tag{24}$$

In Eq. (24) we have rewritten the determinants as sums over all permutations by using the completely antisymmetric Levi-Cevità tensor $\epsilon_{c_1\ldots c_N}$ which is 1 if $c_1,\ldots,c_N$ is an even permutation of $[1,\ldots,N]$, is -1 if it is an odd permutation, and is otherwise zero. As mentioned above, the quantity $(\Delta_N D)$ in the square brackets in Eq. (24) is exponentially convergent to zero when any λ_i becomes large (thus the quantity in the curly brackets is also exponentially convergent). Further, so long as $c_i-1\neq 0$ (which implies $c_i-1+k_{d_i}\neq 0$) the quantity in curly brackets goes to zero at the lower boundary $\lambda_i=0$. This enables us to trivially integrate by parts with respect to λ_i where we differentiate the quantity in the curly brackets and integrate the quantity outside the curly brackets and we do not obtain any boundary terms. We choose to do this integration exactly c_i-1 times to obtain

$$G_\nu(z) = \frac{\mathcal{CN}}{(-z)^{N(N-1)/2}}\sum_{c_1,\ldots,c_N}\epsilon_{c_1\cdots c_N}\int d\boldsymbol{\lambda}\prod_{i=1}^N(\lambda_i - z)^{\nu+N-1} \times(-1)^{c_i-1}\sum_r\Phi_r\sum_{d_1\cdots d_N}\epsilon_{d_1\cdots d_N}\prod_{i=1}^N\lambda_i^{k_{d_i}}\prod_{p=1}^{c_i-1}\frac{k_{d_i}+p}{\nu+N-p}.$$

We would now like to interchange the order of integration and summation such that all integrals are done first. However, if we did this we would end up with divergent integrals. To fix this problem, we insert a cutoff function such as $f(\lambda)=\exp[-\delta\lambda]$ and at the end of the calculation we take δ to zero. (The precise form of the cutoff function will not matter.) This allows us to reorder and write

$$G_\nu(z) = \frac{\mathcal{CN}}{(-z)^{N(N-1)/2}}\sum_r\Phi_r\sum_{c_1\cdots c_N}\epsilon_{c_1\cdots c_N}\sum_{d_1\cdots d_N}\epsilon_{d_1\cdots d_N} \times\prod_{i=1}^n(-1)^{c_i-1}\left[\int_0^\infty d\lambda_i f(\lambda_i)(\lambda_i - z)^{\nu+N-1}\lambda_i^{k_{d_i}} \times\prod_{p=1}^{c_i-1}\frac{k_{d_i}+p}{\nu+N-p}\right].$$

We can now do the sums over c's and d's to obtain

$$G_\nu(z) = N!\,\mathcal{CN}\,z^{-N(N-1)/2}\sum_r\Phi_r\det[P_{ij}^{(r)}], \tag{25}$$

$$P_{ij}^{(r)} = P_{i1}^{(r)}\prod_{p=1}^{j-1}\frac{k_i+p}{\nu+N-p}, \tag{26}$$

$$P_{i1}^{(r)} = \int_0^\infty d\lambda\, f(\lambda)(\lambda - z)^{\nu+N-1}\lambda^{k_i}. \tag{27}$$

The rather special form of the matrix expressed in Eq. (26) allows us to calculate the determinant straightforwardly yielding $\det[P_{ij}^{(r)}]=\Delta_N(k)J_\nu^{-1}\Pi_{i=1}^N P_{i1}^{(r)}$. Thus we have

$$\Phi_r\det[P_{ij}^{(r)}] = \frac{F_N^2\det[a_i^{k_j}]\det[b_i^{k_j}]\prod_{i=1}^N[(-1)^{k_i}/k_i!]P_{i1}^{(r)}}{(-1)^{N(N-1)/2}\Delta_N(a)\Delta_N(b)J_\nu},$$

where we have used Eqs. (9), (19), (20), and (22) and the definitions of $C(k)$ and $s(k)$. Substituting this result into Eq. (25) we now need only do the sum over r. This sum, as explained above, is actually a sum over $0\leqslant k_N<k_{N-1}\ldots<k_1$. Thus we have

$$G_\nu(z) = S\sum_{0\leqslant k_N\ldots<k_1}\det[a_i^{k_j}]\det[b_i^{k_j}]\prod_{i=1}^N w(k_i), \tag{28}$$

$$w(k) = \frac{(-1)^k}{k!}\int_0^\infty d\lambda\, f(\lambda)(\lambda - z)^{\nu+N-1}\lambda^k, \tag{29}$$

where $S=\mathcal{NCQ}_\nu(z)$ and we have absorbed the numerical constants F_N into $\mathcal{C}$. We can now address the sum in Eq. (28) using the Cauchy-Binet theorem (see the Appendix) to obtain

$G_\nu(z)=S\,\det[W(a_ib_j)]$ with the function $W(x)$ defined by

$$W(x)=\sum_{k=0}^{\infty}x^k w(k)=\int_0^\infty d\lambda\ f(\lambda)(\lambda-z)^{\nu+N-1}e^{-x\lambda}.$$

We can now remove the convergence function f (letting $\delta\to 0$ as discussed above) to obtain [see Eq. (12)] $W(x)\to g(x,\nu+N,z)x^{-N}$. The factors of x^{-N} precisely cancel the prefactor $\mathcal{N}$ and we recover the desired result Eq. (7) for $N=N'$ (where $R_\nu=1$) up to the N-dependent normalization prefactor $\mathcal{C}$ which we have not kept track of. To show that the normalization of Eq. (7) (i.e., $\mathcal{C}=1$) is indeed correct we need only verify that $G_0(z)=1$. To do this [using Eq. (13)] we need to establish

$$\det\left[(N-1)!\sum_{m=0}^{N-1}\frac{(-za_ib_j)^m}{m!}\right]=Q(z,0)^{-1},$$

which is easy to show using the Cauchy-Binet theorem. [In Eq. (A1), use $w(k)=(-z)^k(N-1)!/k!$ for $k\leq N-1$ and $w(k)=0$ otherwise, so the determinants on the left hand side of Eq. (A1) are precisely $\Delta(a)$ and $\Delta(b)$.] This completes the proof for the case of $N=N'$.

Using the results we have derived for square matrices we can now easily derive results for rectangular matrices ($N>N'$). Given an N'-dimensional matrix $\mathbf{B}$ with eigenvalues $b_1,\ldots,b_{N'}$ we consider an auxiliary N-dimensional matrix $\tilde{\mathbf{B}}$ with the N' eigenvalues $b_1,\ldots,b_{N'}$ plus $N-N'$ eigenvalues $b_{N'+1},\ldots,b_N$. We then take a limit where $b_{N'+1},\ldots,b_N$ all go to infinity. By viewing the matrix $\mathbf{M}$ as being $\mathbf{A}^{-1/2}\mathbf{Z}\,\tilde{\mathbf{B}}^{-1/2}$ it is clear that taking this limit drives $N-N'$ columns of $\mathbf{M}$ to zero and we obtain effectively an ($N\times N'$)-dimensional problem (with $N-N'$ additional zero eigenvalues). To take these limits we will use the expansion (Ref. [14], Eq. 8.357)

$$\lim_{x\to\infty}g(x,N+\nu,z)=x^{N-1}(-z)^{N+\nu-1}\left[1+\frac{N+\nu-1}{(-zx)}+\frac{(N+\nu-1)(N+\nu-2)}{(-zx)^2}+\cdots\right].\qquad(30)$$

As $b_N\to\infty$ we use the first term of this expansion and replace the $j=N$ row of the matrix L_{ij} in Eq. (7) with $(a_ib_N)^{N-1}(-z)^{N+\nu-1}$. In the denominator ($Q_\nu$) we have $\Delta_N(b)\to(b_N)^{N-1}\Delta_{N-1}(b)$ so the factors of b_N cancel to give a finite ratio. We next let $b_{N-1}\to\infty$. In taking this limit the first term in the expansion Eq. (30) would result in the $j=N-1$ row of L_{ij} being exactly proportional to the $j=N$ row and thus we would obtain $\det L_{ij}=0$. Thus, the leading divergence as $b_{N-1}\to\infty$ is actually from the second term of the expansion (30). We can then replace the $j=N-1$ row of L_{ij} with $(N+\nu-1)(a_ib_{N-1})^{N-2}(-z)^{N+\nu-2}$. Again, the diverging powers of b_{N-1} here are canceled by powers in the denominator (Q_ν) since $\Delta_{N-2}(b)=(b_{N-1})^{N-2}\Delta_{N-1}(b)$. This procedure can be continued until we have let all $b_{N'+1}\cdots b_N\to\infty$. We cancel all of the diverging terms, then factor out the numerical prefactors (such as $N+\nu-1$) to give R_ν, and factor out common factors of $-z$ to obtain the general result quoted above in Eqs. (7)–(11) times $(-z)^{\nu(N-N')}$, which is due to the fact that, as mentioned above, our auxiliary problem has $N-N'$ zero eigenvalues. In this way we complete our more general proof.

APPENDIX: CAUCHY-BINET THEOREM

Given N-dimensional vectors a_i and b_i, and a function $W(z)=\sum_{i=0}^{\infty}w(i)z^i$ convergent for $|z|<\rho$ then if $|a_ib_j|<\rho$ for all i,j we have (where the determinants are all taken with respect to the indices i and j) [13]

$$\sum_{0\leq k_N\cdots<k_1}\det[a_i^{k_j}]\det[b_i^{k_j}]\prod_{i=1}^{N}w(k_i)=\det[W(a_ib_j)].\quad\text{(A1)}$$

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